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A NOTE ON THE NECESSARY CONDITION
FOR LINEAR SHARING AND SEPARATION[†]

by

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and
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WP #1571-84

September 1983

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[†] After this note was written we learned that the result here is similar to one of Ross [5] but in a different context.

1. Introduction and Conclusion

A necessary and sufficient condition under which an optimal sharing rule is linear, as generally accepted by the finance community, is that utility functions be of the equicautious HARA class. (See Mossin [4] for a discussion.) As pointed out by Amershi and Stoeckenius [1] for a given weighting $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ on agents' utilities, equicautious HARA class is indeed a necessary condition when agents have heterogeneous beliefs. When agents have homogeneous beliefs, however, the equicautious HARA class of utilities is not a necessary condition. This last point can easily be seen from the following trivial example: Let there be two identical individuals with weighting (1,1). Then the optimal sharing rule is that both get one-half of the aggregate endowment in each and every state, which is clearly linear. However, their common utility function is arbitrary, not necessarily of the equicautious HARA class.

Note that in the above example, if the weighting $\lambda = (\lambda_1, \lambda_2)$ is such that $\lambda_1 \neq \lambda_2$, then the optimal sharing rule is not linear. Implicit in Mossin's necessity part derivation is the assumption that for all weightings λ in an open set the Pareto-optimal sharing rules are linear (cf. [4], pp. 114). Thus it is not surprising that, for a fixed weighting λ , for the optimal sharing rule to be linear it is not necessary that utilities be of the equicautious HARA class. Furthermore, since the fund separation property for a given λ ¹ and for some endowment whose range is R is equivalent to a linear optimal sharing rule for this given λ . Thus equicautious HARA utilities are not necessary for fund separation with respect to a given λ either. The equicautious HARA utilities are necessary, however, if fund separation is to obtain for all λ (cf. Amershi and Stoeckenius [1], Corollary 17).²

The purpose of this short note is therefore to derive a necessary (and sufficient) condition, such that the optimal sharing rule is linear and

separation obtains, for a fixed weighting λ . This condition requires that agents' cautiousness be identical (but not necessarily determinate) along the optimal schedule in contrast to being identical and determinate in the equicautions HARA case. We also demonstrate how easy it is to construct utility functions, for a given weighting λ , such that the condition we characterize is satisfied.

2. The Results

Instead of going through the formality, for which we refer readers to Amershi and Stoeckenius [1] or Wilson [7], we will speak rather loosely. Let there be N agents in the economy, indexed by $i = 1, 2, \dots, N$. Each agent is characterized by an increasing and strictly concave von Neumann-Morgenstern utility function U_i . It is assumed that agents have homogeneous beliefs on the state space S , the generic element of which is denoted by s . Let $w: S \rightarrow R$ denote the aggregate endowment in the economy. We assume that the range of w is R . A sharing rule is a vector-valued function $z = (z_1, z_2, \dots, z_N): R \rightarrow R^N$ such that

$$\sum_{i=1}^N z_i(x) = x \quad \forall x \in R$$

A sharing rule z is said to be Pareto-optimal if there is no other sharing rule z' such that

$$E(U_i(z_i'(w(s)))) \geq E(U_i(z_i(w(s)))) \quad \forall i$$

and with at least a strict inequality for some i . There are an infinite number of Pareto-optimal sharing rules.

Remark: Here we note that by defining a sharing rule to be a vector-valued function on R rather than a vector-valued function on $R \times S$ is without loss of generality. When agents have homogeneous beliefs, optimal sharing rules are state independent (cf. Wilson [6]).

Now we first record some results due to Borch [2]:

- (a) A sharing rule z is Pareto-optimal if and only if there exists constants (weighting) $(\lambda_1, \lambda_2, \dots, \lambda_N) = \lambda$ such that

$$\forall i, j \quad \lambda_i U_i'(z_i(x)) = \lambda_j U_j'(z_j(x)) \quad \text{for all } x \in R.$$

$$(b) \quad z_i'(x) = \frac{\rho_i(z_i(x))}{\rho_0(x)}, \text{ where } \rho_i(z_i(x)) = -u_i'(z_i(x))/u_i^{ii}(z_i(x))$$

is agent i 's risk tolerance, and where $\rho_0(x) = \sum_{i=1}^N \rho_i(z_i(x))$.

The cautiousness $\sigma_i(z_i(x))$ is defined to be

$$\sigma_i(z_i(x)) = \rho_i'(z_i(x))$$

Our main result is:

PROPOSITION: Let z be a Pareto-optimal sharing rule. Then z is linear iff

$$\sigma_i(z_i(x)) = \sigma_j(z_j(x)) \quad \forall i, j$$

PROOF: The sharing rule z is linear iff $z_i^{ii}(x) = 0 \quad \forall i$ and $\forall x \in R$. This is equivalent to

$$\frac{d}{dx} \left(\frac{\rho_i(z_i(x))}{\rho_0(x)} \right) = 0, \quad \forall i \quad \forall x \in R$$

by (b). Carrying out differentiation yields:

$$\rho_0(x)\sigma_i(z_i(x))z_i'(x) = \rho_i(z_i(x))\rho_0'(x) \quad \forall i \quad \forall x \in R.$$

Substituting (b) for $z_i'(x)$ gives:

$$\sigma_i(z_i(x)) = \rho_0'(x) \quad \forall i \quad \forall x \in R.$$

Thus we have:

$$\sigma_i(z_i(x)) = \sigma_j(z_j(x)) \quad \forall i, j, \forall x \in R.$$

Q.E.D.

Interpreting the above result in an exchange economic setting, the necessary and sufficient condition for the linear sharing rule to be Pareto-optimal and for fund separation to obtain is a joint condition on both agents' utilities and the distribution of initial endowments. The latter is captured by the weighting λ associated with the optimal sharing rule. Examples other than the

trivial one in the Introduction can easily be constructed to demonstrate the above Proposition: Given a weighting (λ_1, λ_2) , two scalars a and $0 < b < 1$, and an increasing strictly concave utility function $U_1(\cdot)$, we can construct another utility function $U_2(\cdot)$ increasing and strictly concave such that

$$\lambda_1 U_1'(a + bx) = \lambda_2 U_2'(-a + (1-b)x) \quad \forall x \in \mathbb{R}.$$

This is easily done by defining

$$U_2(y) = c + \frac{\lambda_1}{\lambda_2} \int_0^y U_1' \left(\frac{a+bt}{1-b} \right) dt \quad \forall y \in \mathbb{R},$$

where c is an arbitrary scalar. Then under the weighting (λ_1, λ_2) , the optimal sharing rule for the group $\langle U_1, U_2 \rangle$ is $\langle a + bx, -a + (1-b)x \rangle$, clearly linear. And of course, the identical cautiousness condition as required of the Proposition is satisfied.

Finally we should note that either when agents' beliefs are heterogeneous or when the optimal sharing rules are linear over an open set of λ , we have weak aggregation (cf. Rubinstein [6]). That is, equilibrium prices are stable with respect to a perturbation of the weighting (or initial distribution of endowments). Under the condition of the Proposition, although the optimal sharing rule is linear, equilibrium prices are not stable with respect to a change of λ .

FOOTNOTES

1. Here we mean for the equilibrium corresponding to the given weighting λ , all individuals hold varying proportions of a riskless asset and a single risky portfolio, which is the same for all individuals.
2. Cass and Stiglitz [3] demonstrated that for monetary separation to obtain for an individual at varying levels of initial wealth, it is necessary that his utility function be of HARA type. When we are considering a group of individuals, varying the weighting λ is just like varying the distribution of wealth in a market economy and therefore the levels of initial wealth. Thus for each individual in the group to obtain monetary separation when varying λ , it is necessary that his utility function is of HARA type. For the group as a whole to exhibit separation property for an open set of λ , however, agents' utility functions should be of equi-cautious HARA class.

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